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Exact Dynamic Analysis of Space Structures Using Timoshenko Beam Theory

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A method employing an exact transfer dynamic stiffness matrix (TDSM) based on Timoshenko beam theory for dynamic analysis of three-dimensional frame structures was developed, where TDSM was a frequency-dependent basis function. In this method, the TDSM of each member was assembled to get the system matrix. All free-vibration solutions, including repeated roots for the characteristic equation, could be obtained to any desired accuracy using an equation developed by Wittrick and Williams. The exact mode shape of the structures can then be computed using the dynamic shape function and the eigenvector. The accuracy of the present method is demonstrated by two examples. The results show good agreement with those computed by the finite element method.

Nomenclature

A	=	cross-sectional area
C	=	unknown coefficient vector
E	=	Young's modulus
G	=	shear modulus
I_p	=	mass polar moment of inertia of the shaft per unit length
I_z, I_y	=	moment of inertia about z and y axes
J	=	number of eigenvalues below the trial frequency ω_t
J_t	=	torsional constant depended on the shape of cross section
J_0	=	number of eigenvalues of the structure lying between zero and the trial frequency when the displacement components corresponding to $\boldsymbol{K}_R$ are all fixed
$\boldsymbol{K}$	=	transfer dynamic stiffness matrix for a three-dimensional beam member
$\boldsymbol{K}_G$	=	global transfer dynamic stiffness matrix
$\boldsymbol{K}_R$	=	reduced global dynamic stiffness matrix
$\boldsymbol{K}_R^U$	=	upper triangular matrix obtained by using lower–upper factorization
k	=	Timoshenko's shear coefficient
l	=	length of member
M	=	moment
$\boldsymbol{N}(\boldsymbol{x})$	=	dynamic shape function
$\boldsymbol{S}_x$	=	states at point x
$\boldsymbol{S}_0$	=	states at $x = 0$
$s\{\boldsymbol{K}_R^U(\omega_t)\}$	=	number of negative elements on the leading diagonal of $\boldsymbol{K}_R^U$
$\boldsymbol{T}$	=	transfer matrix
$\boldsymbol{U}$	=	displacement states at a point in a member
$\boldsymbol{U}_a$	=	displacement at end a
$\boldsymbol{U}_b$	=	displacement at end b
$\boldsymbol{U}^e$	=	end displacement
$\boldsymbol{U}_R$	=	corresponding eigenvector
V	=	shear force
w	=	deformation
θ	=	slope
ρ	=	mass density
ω_n	=	circular frequency
ω_t	=	trial frequency

Subscripts

d, f	=	displacement and force states
x, y, z	=	directions in axes x, y, and z

I. Introduction

COMPLEX and large frame structures are frequently used in the design of space structures such as space stations and deployable antenna systems, owing to their low cost, light weight, and high stiffness, as well as ease of packing, transporting, and assembling. These structures may experience dynamic loading. Therefore, the vibration characteristics of the aforementioned kinds of frame structures must be predicted accurately. For a general framed structure made up of beam members, the finite element method (FEM) is usually used to obtain an approximate solution of the vibration problem. The accuracy of the solution depends on the finite element mesh used in the model. Alternatively, one can use a continuum model to find the exact frequencies. The continuum model can be formed using classical Euler–Bernoulli beam theory. The governing equations for vibration analysis of a three-dimensional frame based on beam theory constitute a system of partial differential equations. One can find the natural frequencies by solving these equations by applying the boundary and internal compatibility conditions to these partial differential equations. In general, this frequency equation is a transcendental equation that yields an infinite number of natural frequencies and normal modes. The classical Bernoulli–Euler theory of flexural vibration is characterized by giving higher natural frequencies than those obtained by experiments on the thick beams, especially for higher modes. To improve the beam theory, Rayleigh[1]

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improved the classical theory by considering the effect of the rotary inertia of the cross section, and Timoshenko[2,3] introduced the effect of transverse shear deformation.

A popular method for solving the structure member vibration problem is the transfer matrix method.[4] This method has a long history and was first used by Myklestad,[5,6] who presented a tabular method for finding natural frequencies. Pestel and Leckie[4] reformed the tabular method as a transfer matrix method. It has been successfully applied to simple structures in which one geometric dimension is predominant with respect to the others (beams, shafts). It allows the analysis of the structure to be effected by the subsequent translation of the characteristic section state variables, including displacement, angle of rotation, shear force, and bending moment from the initial section of the system to the final one. The relationship is established between a set of unknowns on the two ends of the structure section. The transfer matrix method (TMM) can also be extended to solve the three-dimensional frame vibration problem. However, the TMM has the following disadvantages: 1) The matrix is not symmetric. 2) Because of the mixing of force and displacement states, the entries in the transfer matrix have a wide range of values. Thus, we chose to use the equivalent dynamic stiffness formulation.

The dynamic stiffness matrix (DSM) method can be considered an improved finite element model because the element DSM is exact. Several algorithms based on the DSM used to find the nontrivial solutions are presented. Banerjee[7] used the DSM method to obtain the eigenvalue while the frequency determinant is zero. Moon and Choi[8] presented a sign function Z to obtain the true roots by applying a bisection method to this sign function because the bisection method necessitates only the sign of the value of the function. The exact method of computing natural frequency using the DSM method in the literature involved difficulty with the poles because the eigenvalues were computed by the frequency determinant. In addition, it is difficult to obtain all of the eigenvalues by the searching method when there are repeated roots or when the eigenvalues are too close to each other. Also, the mode shape presented was only based on FEM and not the exact beam theory.

Hence, the purpose of this paper is to develop a method for exact vibration analysis of a three-dimensional frame using the exact transfer dynamic stiffness matrix (TDSM) based on the Timoshenko beam theory.[3] All of the natural frequencies, including repeated eigenvalues, can be obtained by using TDSM with Wittrick and Williams's equation.[9] The accurate dynamic characteristics of the structures are determined by the exact dynamic shape function and the eigenvector.

II. Derivation of the TDSM

Given a three-dimensional frame consisting of N prismatic members, the vibration analysis of each member involves the solution of four differential equations. These are axial vibration, torsional vibration, and flexural vibration in two planes. Thus, the vibration analysis of this frame involves the solution of 4 N differential equations. A general procedure for deriving the transfer dynamic matrix is presented in this section. For a prismatic member, each point on the beam has six degrees of freedom (DOF). These DOF are divided into axial motion along the beam, torsional motion about the beam axis, and bending in two planes. There is a force state corresponding to each DOF.

The sign convection for positive end forces and displacements defined in the transfer matrix are shown in Fig. 1, and the states for the vibration mode are defined in Table 1. In the following, we first summarize the equation of motion for each mode of vibration. Then we express the solution as functions of sets of unknown coefficients. Next, we derive the transfer matrix for the beam section by expressing these unknown coefficients in terms of the states at one end. Finally, TDSMs are derived by rearranging the elements in the transfer matrices.

A. Equation of Motion

By the use of the Timoshenko beam theory, the equation of motion for torsional, axial, and flexural vibrations in two planes can be expressed as follows. For flexural vibration in the x–y plane,

Table 1 Definition of states for vibration mode

	State	
Vibration mode	Displacement	Force
Axial	w_x	V_x
Torsional	θ_x	M_x
Bending in x–y plane	$w_y,\ \theta_z$	$V_y,\ M_z$
Bending in x–z plane	$w_z,\ \theta_y$	$V_z,\ M_y$

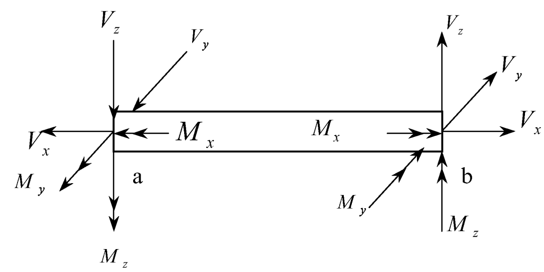

Fig. 1a End forces.

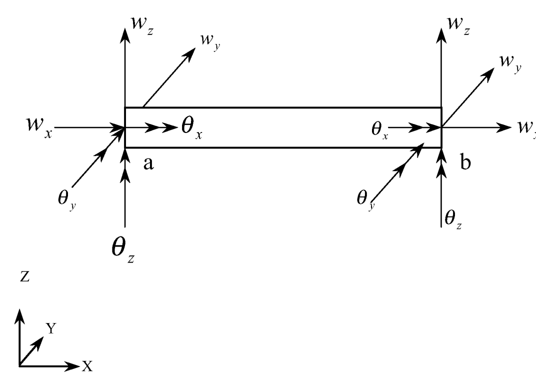

Fig. 1b Displacements.

$$\frac{d^4 w_y}{dx^4} + \left(\frac{\sigma + \tau_{xy}}{l^2}\right)\frac{d^2 w_y}{dx^2} - \left(\frac{\beta_{xy}^4 - \sigma\tau_{xy}}{l^4}\right) w_y = 0 \quad (1)$$

For flexural vibration in the x–z plane,

$$\frac{d^4 w_z}{dx^4} + \left(\frac{\sigma + \tau_{xz}}{l^2}\right)\frac{d^2 w_z}{dx^2} - \left(\frac{\beta_{xz}^4 - \sigma\tau_{xz}}{l^4}\right) w_z = 0 \quad (2)$$

For axial vibration in the x direction,

$$\frac{d^2 w_x}{dx^2} + \beta_e^2 w_x = 0 \quad (3)$$

For torsional vibration within the x direction,

$$\frac{d^2 \theta_x}{dx^2} + \beta_t^2 \theta_x = 0 \quad (4)$$

where

$$\mu = \rho A, \qquad A_s = kA, \qquad \sigma = \frac{\mu\omega^2}{GA_s} l^2$$

$$r_z = \sqrt{\frac{I_z}{A}}, \qquad \tau_{xy} = \frac{\mu r_z^2 \omega^2}{EI_z} l^2, \qquad \beta_{xy}^4 = \frac{\mu\omega^2}{EI_z} l^4$$

$$r_y = \sqrt{\frac{I_y}{A}}, \qquad \tau_{xz} = \frac{\mu r_y^2 \omega^2}{EI_z} l^2, \qquad \beta_{xz}^4 = \frac{\mu\omega^2}{EI_z} l^4$$

$$\beta_e^2 = \frac{\omega^2}{(E/\rho)}, \qquad \beta_t^2 = \frac{\omega^2}{\left(GJ_t/\rho I_p\right)}$$

The general solutions of $w_y(x)$, $\theta_z(x)$, $V_y(x)$, and $M_z(x)$ for free transverse vibration in the x–y plane are

$$w_y(x) = -\frac{l^4}{\beta^4 EI_z}\left[A_1\frac{\lambda_1}{l}\sinh\left(\lambda_1\frac{x}{l}\right) + A_2\frac{\lambda_1}{l}\cosh\left(\lambda_1\frac{x}{l}\right) - A_3\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right) + A_4\frac{\lambda_2}{l}\cos\left(\lambda_2\frac{x}{l}\right)\right] \tag{5}$$

$$\theta_z(x) = \frac{-l^2}{\beta^4 EI_z}\left\{\left(\sigma+\lambda_1^2\right)\left[A_1\cosh\left(\lambda_1\frac{x}{l}\right) + A_2\sinh\left(\lambda_1\frac{x}{l}\right)\right] + \left(\sigma-\lambda_2^2\right)\left[A_3\cos\left(\lambda_2\frac{x}{l}\right) + A_4\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right)\right]\right. \tag{6}$$

$$V_y(x) = A_1\cosh\left(\lambda_1\frac{x}{l}\right) + A_2\sinh\left(\lambda_1\frac{x}{l}\right) + A_3\cos\left(\lambda_2\frac{x}{l}\right) + A_4\sin\left(\lambda_2\frac{x}{l}\right) \tag{7}$$

$$M_z(x) = \frac{-l^2}{\beta^4}\left\{\left(\sigma+\lambda_1^2\right)\frac{\lambda_1}{l}\left[A_1\sinh\left(\lambda_1\frac{x}{l}\right) + A_2\cosh\left(\lambda_1\frac{x}{l}\right)\right] - \left(\sigma-\lambda_2^2\right)\frac{\lambda_2}{l}\left[A_3\sin\left(\lambda_2\frac{x}{l}\right) - A_4\cos\left(\lambda_2\frac{x}{l}\right)\right]\right\} \tag{8}$$

The general solutions of $w_z(x)$, $\theta_y(x)$, $V_z(x)$, and $M_y(x)$ for free transverse vibration in the x–z plane are

$$w_z(x) = -\frac{l^4}{\beta^4 EI_y}\left[B_1\frac{\lambda_1}{l}\sinh\left(\lambda_1\frac{x}{l}\right) + B_2\frac{\lambda_1}{l}\cosh\left(\lambda_1\frac{x}{l}\right) - B_3\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right) + B_4\frac{\lambda_2}{l}\cos\left(\lambda_2\frac{x}{l}\right)\right] \tag{9}$$

$$\theta_y(x) = \frac{l^2}{\beta^4 EI_y}\left\{\left(\sigma+\lambda_1^2\right)\left[B_1\cosh\left(\lambda_1\frac{x}{l}\right) + B_2\sinh\left(\lambda_1\frac{x}{l}\right)\right] + \left(\sigma-\lambda_2^2\right)\left[B_3\cos\left(\lambda_2\frac{x}{l}\right) + B_4\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right)\right]\right\} \tag{10}$$

$$V_z(x) = B_1\cosh\left(\lambda_1\frac{x}{l}\right) + B_2\sinh\left(\lambda_1\frac{x}{l}\right) + B_3\cos\left(\lambda_2\frac{x}{l}\right) + B_4\sin\left(\lambda_2\frac{x}{l}\right) \tag{11}$$

$$M_y(x) = \frac{l^2}{\beta^4}\left\{\left(\sigma+\lambda_1^2\right)\frac{\lambda_1}{l}\left[B_1\sinh\left(\lambda_1\frac{x}{l}\right) + B_2\cosh\left(\lambda_1\frac{x}{l}\right)\right] - \left(\sigma-\lambda_2^2\right)\frac{\lambda_2}{l}\left[B_3\sin\left(\lambda_2\frac{x}{l}\right) - B_4\cos\left(\lambda_2\frac{x}{l}\right)\right]\right\} \tag{12}$$

The general solutions of $w_x(x)$ and $V_x(x)$ for free axial vibration are

$$w_x(x) = C_1\cos\beta_e x + C_2\sin\beta_e x \tag{13}$$

$$V_x = -C_1 EA\beta_e\sin\beta_e x + C_2 EA\beta_e\cos\beta_e x \tag{14}$$

The general solutions of $\theta_x(x)$ and $M_x(x)$ for free torsional vibration are

$$\theta_x(x) = D_1\cos\beta_t x + D_2\sin\beta_t x \tag{15}$$

$$M_x = -D_1 GJ_t\beta_t\sin\beta_t x + D_2 GJ_t\beta_t\cos\beta_t x \tag{16}$$

Now, we can express the force and displacement in a general matrix form as

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = BC \tag{17}$$

where the following apply:

For axial vibration,

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_x \\ V_x \end{Bmatrix}$$

$$B = \begin{bmatrix} \cos\beta_e x & \sin\beta_e x \\ -EA\beta_e\sin\beta_e x & EA\beta_e\cos\beta_e x \end{bmatrix}$$

For torsional vibration,

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} \theta_x \\ M_x \end{Bmatrix}$$

$$B = \begin{bmatrix} \cos\beta_t x & \sin\beta_t x \\ -GJ_t\beta_t\sin\beta_t x & GJ_t\beta_t\cos\beta_t x \end{bmatrix}$$

For flexural vibration in the x–y plane,

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_y \\ \theta_z \\ V_y \\ M_z \end{Bmatrix}$$

$$B = \begin{bmatrix} \frac{-l^3\lambda_1}{\beta^4 EI_z}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^3\lambda_1}{\beta^4 EI_z}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^3\lambda_2}{\beta^4 EI_z}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l^3\lambda_2}{\beta^4 EI_z}\cos\left(\lambda_2\frac{x}{l}\right) \\ \frac{-l^2\left(\sigma+\lambda_1^2\right)}{\beta^4 EI_z}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^2\left(\sigma+\lambda_1^2\right)}{\beta^4 EI_z}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^2\left(\sigma-\lambda_2^2\right)}{\beta^4 EI_z}\cos\left(\lambda_2\frac{x}{l}\right) & \frac{-l^2\left(\sigma-\lambda_2^2\right)}{\beta^4 EI_z}\sin\left(\lambda_2\frac{x}{l}\right) \\ \cosh\left(\lambda_1\frac{x}{l}\right) & \sinh\left(\lambda_1\frac{x}{l}\right) & \cos\left(\lambda_2\frac{x}{l}\right) & \sin\left(\lambda_2\frac{x}{l}\right) \\ \frac{-l\left(\sigma+\lambda_1^2\right)\lambda_1}{\beta^4}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l\left(\sigma+\lambda_1^2\right)\lambda_1}{\beta^4}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l\left(\sigma-\lambda_2^2\right)\lambda_2}{\beta^4}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l\left(\sigma-\lambda_2^2\right)\lambda_2}{\beta^4}\cos\left(\lambda_2\frac{x}{l}\right) \end{bmatrix}$$

For flexural vibration in the x–z plane,

$$\begin{Bmatrix} d \\ f \end{Bmatrix} = \begin{Bmatrix} w_z \\ \theta_y \\ V_z \\ M_y \end{Bmatrix}$$

$$B = \begin{bmatrix} \frac{-l^3\lambda_1}{\beta^4 EI_y}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{-l^3\lambda_1}{\beta^4 EI_y}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^3\lambda_2}{\beta^4 EI_y}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{-l^3\lambda_2}{\beta^4 EI_y}\cos\left(\lambda_2\frac{x}{l}\right) \\ \frac{l^2(\sigma+\lambda_1^2)}{\beta^4 EI_y}\cosh\left(\lambda_1\frac{x}{l}\right) & \frac{l^2(\sigma+\lambda_1^2)}{\beta^4 EI_y}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{l^2(\sigma-\lambda_2^2)}{\beta^4 EI_y}\cos\left(\lambda_2\frac{x}{l}\right) & \frac{l^2(\sigma-\lambda_2^2)}{\beta^4 EI_y}\sin\left(\lambda_2\frac{x}{l}\right) \\ \cosh\left(\lambda_1\frac{x}{l}\right) & \sinh\left(\lambda_1\frac{x}{l}\right) & \cos\left(\lambda_2\frac{x}{l}\right) & \sin\left(\lambda_2\frac{x}{l}\right) \\ \frac{l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\sinh\left(\lambda_1\frac{x}{l}\right) & \frac{l(\sigma+\lambda_1^2)\lambda_1}{\beta^4}\cosh\left(\lambda_1\frac{x}{l}\right) & -\frac{l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\sin\left(\lambda_2\frac{x}{l}\right) & \frac{l(\sigma-\lambda_2^2)\lambda_2}{\beta^4}\cos\left(\lambda_2\frac{x}{l}\right) \end{bmatrix}$$

Note that C depends on the boundary conditions.

B. Transfer Matrix

The unknown constants in terms of states can be expressed as

$$\boldsymbol{S}_x = \begin{Bmatrix} d_x \\ f_x \end{Bmatrix} = B(x)C \tag{18}$$

At point $x = 0$, Eq. (18) can be obtained as

$$\boldsymbol{S}_0 = \begin{Bmatrix} d_0 \\ f_0 \end{Bmatrix} = B(0)C \tag{19}$$

Thus, the coefficient C can be obtained by

$$C = B(0)^{-1} \cdot \boldsymbol{S}_0 \tag{20}$$

Substitute Eq. (20) into Eq. (18) to yield

$$\boldsymbol{S}_x = B(x)B(0)^{-1} \cdot \boldsymbol{S}_0 = \boldsymbol{T} \cdot \boldsymbol{S}_0 \tag{21}$$

where $\boldsymbol{T}$ can be expressed as

$$\boldsymbol{T} = B(x)B(0)^{-1} \tag{22}$$

C. Derivation of TDSM

The transfer matrix relationships of Eq. (21) can be written as

$$\boldsymbol{S}_x = \boldsymbol{T}(x) \cdot \boldsymbol{S}_0 \tag{23}$$

Equation (23) can be expanded as

$$\boldsymbol{S}_x = \begin{Bmatrix} d_x \\ f_x \end{Bmatrix} = \begin{bmatrix} \boldsymbol{T}_{dd} & \boldsymbol{T}_{df} \\ \boldsymbol{T}_{fd} & \boldsymbol{T}_{ff} \end{bmatrix} \begin{Bmatrix} d_0 \\ f_0 \end{Bmatrix} = \boldsymbol{T}(x) \cdot \boldsymbol{S}_0 \tag{24}$$

From Eq. (24), we can solve for the force states in terms of the displacement states,

$$\begin{Bmatrix} f_0 \\ f_x \end{Bmatrix} = \begin{bmatrix} -\boldsymbol{T}_{df}^{-1}\boldsymbol{T}_{dd} & \boldsymbol{T}_{df}^{-1} \\ \boldsymbol{T}_{fd} - \boldsymbol{T}_{ff}\boldsymbol{T}_{df}^{-1}\boldsymbol{T}_{dd} & \boldsymbol{T}_{ff}\boldsymbol{T}_{df}^{-1} \end{bmatrix} \begin{Bmatrix} d_0 \\ d_x \end{Bmatrix} = \boldsymbol{K} \cdot \begin{Bmatrix} d_0 \\ d_x \end{Bmatrix} \tag{25}$$

Note that $\boldsymbol{K}$ is a 12×12 frequency dependent matrix for a three-dimensional beam member.

III. Eigenvalue Analysis

The global TDSM $\boldsymbol{K}_G$ is obtained by assembling the TDSM for each member in the structure. Note that the size of $\boldsymbol{K}_G$ only corresponds to the number of members in the whole structure, which means that the higher modes can be obtained without increasing the size of $\boldsymbol{K}_G$. However, in FEM, the size of the stiffness matrix will be increased for the computation of higher modes. The known state variables in a structural formulation have to be removed through direct elimination, along with boundary conditions, so that no rigid-body DOF exist. This leads to the reduced global DSM $\boldsymbol{K}_R$. For free vibration, this leads to

$$\boldsymbol{K}_R(\alpha)\boldsymbol{U}_R = 0 \tag{26}$$

Equation (26) is commonly known as an eigenvalue problem.

Suppose that ω_n denotes circular frequency in a vibration problem. Thus, the eigenvalues would correspond to natural frequencies. This equation can be solved using the algorithm developed by Wittrick and Williams.[9] By the use of their method, J is given by

$$J = J_0 + s\left\{\boldsymbol{K}_R^U(\omega_t)\right\} \tag{27}$$

Thus, the eigenvalues can be computed by adoption of an iterative procedure with Eq. (27) as an algorithm. This necessitates a number of modes that are under a required frequency. An efficient method for this has been developed.

IV. Eigenfunction Analysis

Once we have computed a natural frequency from Eq. (26), the corresponding eigenvector $\boldsymbol{U}_R$ can be calculated from the solution of the homogeneous equation. From this vector, together with displacement boundary conditions, we have the displacements at all of the joints. To recover the eigenfunction for each member, we have the following interpolation for each member:

$$\boldsymbol{U} = [\boldsymbol{N}(\boldsymbol{x})] \cdot [\boldsymbol{U}^e] \tag{28}$$

Note that $[\boldsymbol{N}(\boldsymbol{x})]$ can be developed as follows. Recall that Eqs. (5) and (6), the general solutions of deformation and slope for free transverse vibration in the x–y plane, are

$$w_y(x) = -\frac{l^4}{\beta^4 EI_z}\left[C_1\frac{\lambda_1}{l}\sinh\left(\lambda_1\frac{x}{l}\right) + C_2\frac{\lambda_1}{l}\cosh\left(\lambda_1\frac{x}{l}\right) - C_3\frac{\lambda_2}{l}\sin\left(\lambda_2\frac{x}{l}\right) + C_4\frac{\lambda_2}{l}\cos\left(\lambda_2\frac{x}{l}\right)\right] \tag{5}$$

$$\theta_z(x) = \frac{-l^2}{\beta^4 EI_z}\left\{\left(\sigma + \lambda_1^2\right)\left[C_1 \cosh\left(\lambda_1 \frac{x}{l}\right) + C_2 \sinh\left(\lambda_1 \frac{x}{l}\right)\right] + \left(\sigma - \lambda_2^2\right)\left[C_3 \cos\left(\lambda_2 \frac{x}{l}\right) + C_4 \frac{\lambda_2}{l} \sin\left(\lambda_2 \frac{x}{l}\right)\right]\right\} \tag{6}$$

Recall that Eqs. (9) and (10), the general solutions of deformation and slope for free transverse vibration in the x–z plane, are

$$w_z(x) = -\frac{l^4}{\beta^4 EI_y}\left[B_1 \frac{\lambda_1}{l} \sinh\left(\lambda_1 \frac{x}{l}\right) + B_2 \frac{\lambda_1}{l} \cosh\left(\lambda_1 \frac{x}{l}\right) - B_3 \frac{\lambda_2}{l} \sin\left(\lambda_2 \frac{x}{l}\right) + B_4 \frac{\lambda_2}{l} \cos\left(\lambda_2 \frac{x}{l}\right)\right] \tag{9}$$

$$\theta_y(x) = \frac{l^2}{\beta^4 EI_y}\left\{\left(\sigma + \lambda_1^2\right)\left[B_1 \cosh\left(\lambda_1 \frac{x}{l}\right) + B_2 \sinh\left(\lambda_1 \frac{x}{l}\right)\right] + \left(\sigma - \lambda_2^2\right)\left[B_3 \cos\left(\lambda_2 \frac{x}{l}\right) + B_4 \frac{\lambda_2}{l} \sin\left(\lambda_2 \frac{x}{l}\right)\right]\right\} \tag{10}$$

Recall that Eq. (13), the general solution of the deflection for free axial vibration, must be

$$w_x(x) = C_1 \cos\beta x + C_2 \sin\beta x \tag{13}$$

Recall that Eq. (15), the general solution of the slope for free torsional vibration, must be

$$\theta_x(x) = C_1 \cos\beta x + C_2 \sin\beta x \tag{15}$$

In general, the preceding displacement function for a beam element can be expressed as

$$[\boldsymbol{U}] = [D(x)] \cdot C \tag{29}$$

For a typical beam element e, the displacement at both ends can be taken as

$$[\boldsymbol{U}^e] = \begin{bmatrix} \boldsymbol{U}_a \\ \boldsymbol{U}_b \end{bmatrix} = \begin{bmatrix} D(x_a = 0) \\ D(x_b = L) \end{bmatrix} \cdot [C] = [A] \cdot C \tag{30}$$

Solving Eq. (30) for the coefficient, we get

$$[C] = [A]^{-1} \cdot [\boldsymbol{U}^e] \tag{31}$$

Substituting Eq. (31) into Eq. (29), we have

$$[\boldsymbol{U}] = [D(x)] \cdot [A]^{-1} \cdot [\boldsymbol{U}^e] = [N] \cdot [\boldsymbol{U}^e] \tag{32}$$

where N can be expressed as

$$[N] = [D(x)] \cdot [A]^{-1} \tag{33}$$

Thus, the eigenfunction can be determined by the eigenvector and the dynamic shape function. Note that the matrix $[N]$ is similar to the shape function matrix in finite element analysis, except that the elements of this matrix are also a function of frequency. Hence, we call it the dynamic shape function matrix.

V. Numerical Examples

Two numerical examples are used to demonstrate the principle and algorithm just described. They are a two-level portal frame and a three-story space structure. The length, cross-sectional area, and material properties for these two examples are shown in Table 2. All structure members have a solid circular cross section.

Table 2 Material propertiesfor illustrated examples

Parameter	Value
Length, in. (m)	100 (2.54)
Young's modulus E, psi (Pa)	10.3×10^6 (71×10^9)
Shear modulus G, psi (Pa)	3.8×10^6 (26.2×10^9)
Mass density ρ, lb/in.^{3} (N/m^3)	$0.098/386.4$ (26.6×10^3)
Cross-sectional area, in.^{2} (m^2)	4 (2.58×10^{-3})

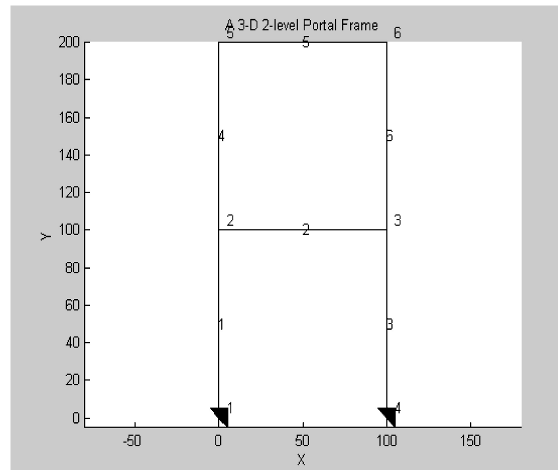

Fig. 2 Two-level portal frame.

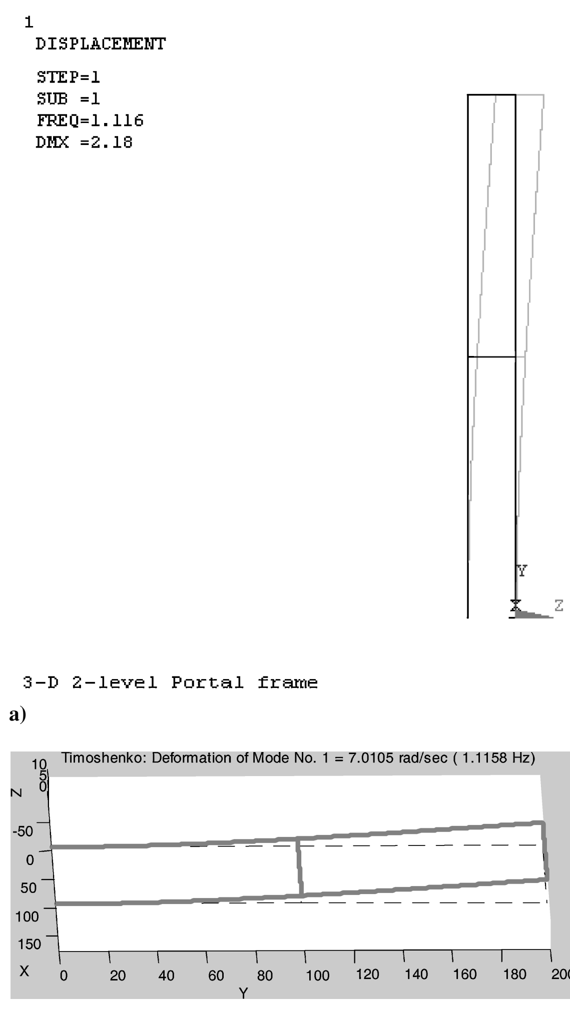

Fig. 3 First mode shape of two-level portal frame by a) FEM and b) exact dynamic shape function.

A. Example 1: Two-Level Portal Frame

There are six nodes and six members in the structure, shown in Fig. 2. The structure is completely fixed at nodes 1 and 4. Table 3 shows the first 10 natural frequencies of the frame by Timoshenko beam theory. In the calculation, the specified accuracy is 10^{-8} for all 10 modes. The results of finite element analysis are also shown in Table 3. The first three mode shapes derived by the dynamic shape function, based on Timoshenko beam theory, are compared with those derived by FEM in Figs. 3–5. The FEM chosen to derive this portal frame by finite element analysis is BEAM4 (ANSYS), which is a uniaxial element with tension, torsion, bending capabilities, and with six DOF at each end. There are 300 elements meshed for each member in the two-level portal frame. Hence, there are a total of 1800 elements meshed when FEM is used to achieve an accurate assessment of this structure. The first mode is an out-of-plane bending mode, the second mode is an in-plane bending mode, and the third one is a twisting mode.

Table 3 Comparison of natural frequency between FEM and Timoshenko beam theory for a two-level portal frame

Mode	FEM (ANSYS)	Timoshenko beam theory	Difference, % [(Exact − FEM)/Exact × 100%]
1	1.1158	1.1158	0
2	2.7735	2.7720	−0.05
3	3.3836	3.3831	−0.01
4	6.6680	6.6655	−0.04
5	9.1055	9.1004	−0.06
6	11.301	11.297	−0.04
7	20.013	19.999	−0.07
8	20.426	20.412	−0.07
9	27.014	26.985	−0.11
10	28.458	28.419	−0.14

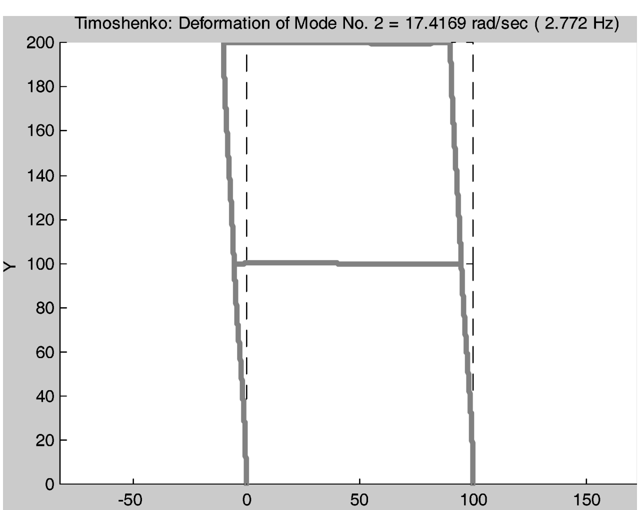

Fig. 4 Second mode shape of two-level portal frame by exact dynamic shape function.

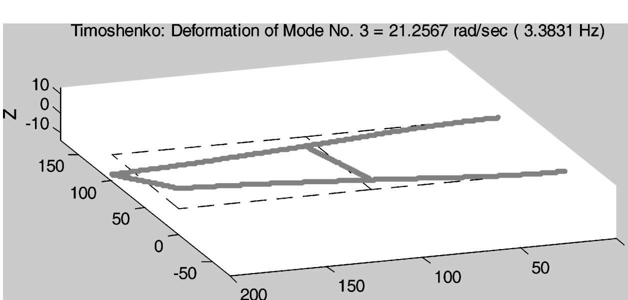

Fig. 5 Third mode shape of two-level portal frame by exact dynamic shape function.

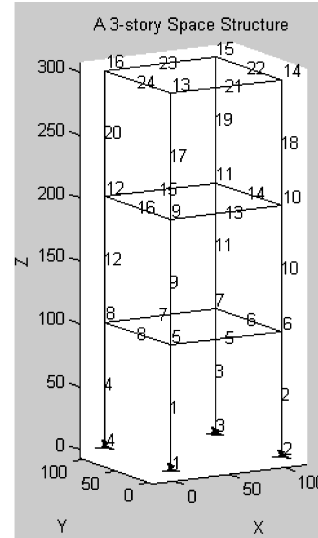

Fig. 6 Three-story space structure.

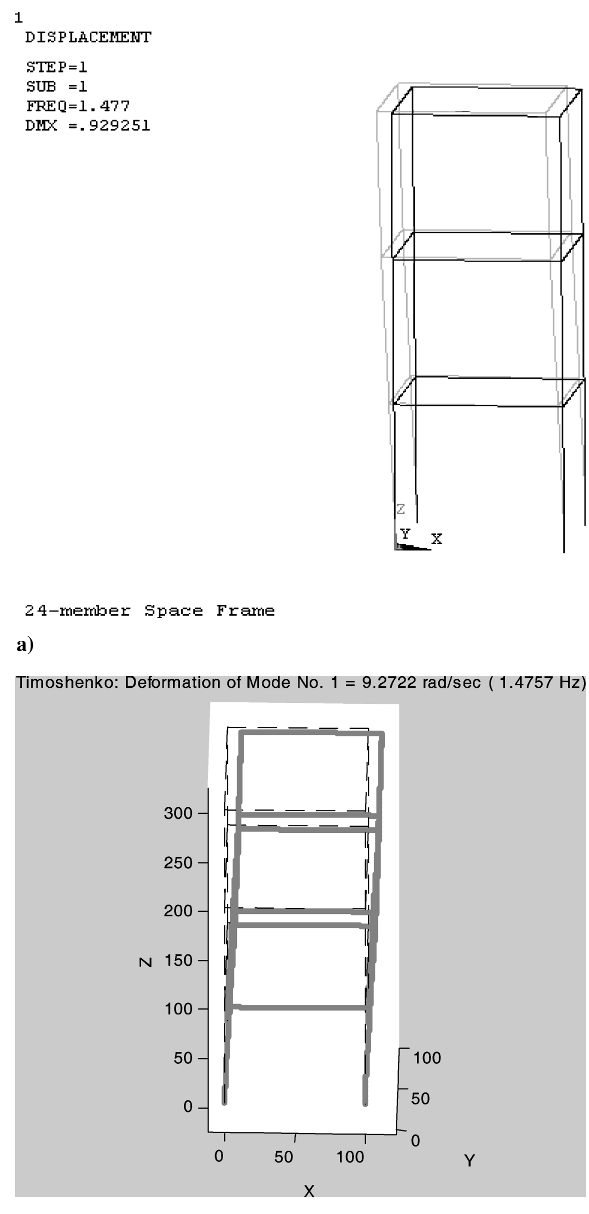

Fig. 7 First mode shape of three-story space structure by a) FEM and b) exact dynamic shape function.

B. Example 2: Three-Story Space Structure

There are 16 nodes and 24 members involved in the three-story frame structure shown in Fig. 6. The structure is completely fixed at nodes 1, 2, 3, and 4. Table 4 shows the first 10 natural frequencies of this space structure derived by Timoshenko beam theory. In the calculation, the specified accuracy is 10^{-8} for all 10 modes. The results of finite element analysis are also shown in Table 4. The first three mode shapes by dynamic shape function based on Timoshenko beam theory are compared with that derived by FEM, as shown in Figs. 7–9. The FEM chosen for the three-story space frame derived with finite element analysis is BEAM4 of ANSYS. There are 75 elements meshed for each member in the three-story space structure. Hence there are a total of 3000 elements meshed when FEM is used to achieve an accurate assessment of this structure. The first mode is a bending mode, the second mode is a repeated bending mode, and the third one is a twisting mode. Because of symmetry, this model is expected to have some repeated natural frequencies. They have been identified by the FEM (Table 4).

Table 4 Comparison of natural frequency between FEM and Timoshenko beam theory for a three-story space structure

Mode	FEM (ANSYS)	Timoshenko beam theory	Difference, % [(Exact − FEM)/Exact × 100%]
1	1.4765	1.4757	−0.05
2	1.4765	1.4757	−0.05
3	1.9376	1.9368	−0.04
4	4.8133	4.8106	−0.06
5	4.8133	4.8106	−0.06
6	6.0776	6.0748	−0.05
7	7.4942	7.4898	−0.06
8	8.4968	8.4919	−0.06
9	8.4968	8.4919	−0.06
10	9.5484	9.5430	−0.06

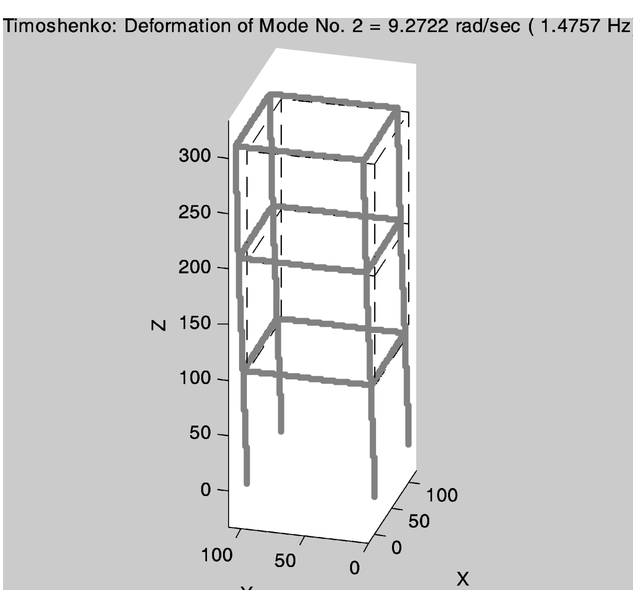

Fig. 8 Second mode shape of three-story space structure by exact dynamic shape function.

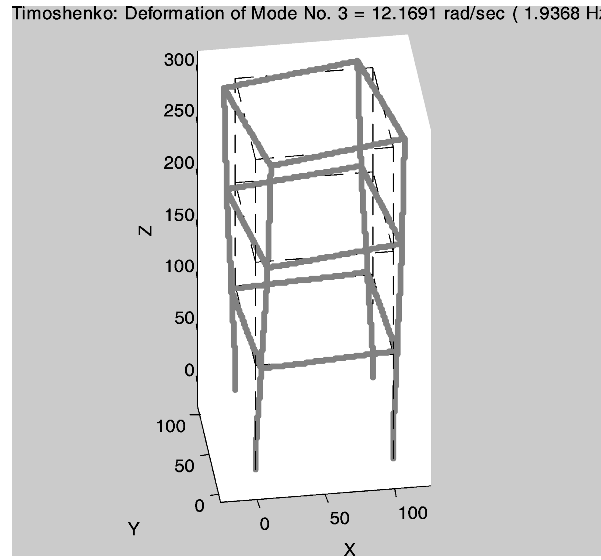

Fig. 9 Third mode shape of three-story space structure by exact dynamic shape function.

VI. Conclusions

A TDSM method based on Timoshenko beam theory for vibration analysis of three-dimensional frame structures was developed in this paper. By the use of the present approach, a vibration solution can be calculated based on exact theory. Numerical examples demonstrated that this method could be used to compute all frequencies, including repeated frequencies, with excellent accuracy compared with those computed by FEM. The mode shape determined by the dynamic shape function and eigenvectors also agreed with that by FEM. We may also use the present approach to validate solutions obtained by FEM.

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